Letting $\tau \equiv \tan\theta$ and $\gamma \equiv \cot\varphi$, (3) and (4) may be transformed to

$$(gr^2 - 2hv^2)\tau^2 + 2rv^2\tau + gr^2 = 0,$$

$$gr^2\gamma^2 - 2rv^2\gamma - (2hv^2 - gr^2) = 0.$$

Hence, under the conditions of the problem

$$\tau = \frac{r(v^2 + R)}{2hv^2 - gr^2}, \qquad \gamma = \frac{v^2 + R}{gr},$$

where

$$R \equiv + \sqrt{v^4 + 2ghv^2 - g^2r^2}.$$

Consequently

$$\cos\theta = \frac{2hv^2 - gr^2}{v\sqrt{2(2h^2v^2 - ghr^2 + r^2v^2 + r^2R)}}, \tag{5}$$

$$\cos\varphi = \frac{v^2 + R}{v\sqrt{2(gh + v^2 + R)}}, \tag{6}$$

By considering solid angles, it may be readily seen that the mass which is deposited on the circular area of radius r, corresponding to the cone of half-angle θ, is given by $\frac{1}{2}m(1 - \cos\theta) \equiv m_\theta$. Similarly the mass issuing from the upper or φ cone and falling on the same area equals $\frac{1}{2}m(1 - \cos\varphi) \equiv m_\phi$. The total mass on the area πr^2 is, therefore, $m_\theta + m_\phi \equiv m_r$. The total mass distributed over the ring $2\pi r dr$ is $(dm_r/dr)dr$, so that the surface density is given by

$$\sigma_r = \frac{1}{2\pi r}\left(\frac{dm_\theta}{dr} + \frac{dm_\phi}{dr}\right). \tag{7}$$

Assuming $v^4 + 2ghv^2 - g^2r^2 \neq 0$ the derivatives involved in (7) may be formed by the aid of (5) and (6), so that, in general

$$\sigma_r = \frac{m}{4\pi vR}\left\{\frac{(gr^2 + 2hR)(v^2 + R)^2}{[2(2h^2v^2 - ghr^2 + r^2v^2 + r^2R)]^{\frac{3}{2}}} + \frac{g^2(2gh + v^2 + R)}{[2(gh + v^2 + R)]^{\frac{3}{2}}}\right\}.$$

Special cases:

(*a*) When $R = 0$, that is, when r attains its superior limit $v/g\sqrt{v^2 + 2gh}$ the θ and φ trajectories coincide, the angles θ and φ become supplementary, and $\varphi = \tan^{-1}(1/v\sqrt{v^2 + 2gh})$.

(*b*) Taking $g = 32$ ft./sec.², $h = 5{,}000$ ft., $v = 1{,}024$ ft./sec., the maximum value of r equals $4{,}608\sqrt{66} \doteq 37{,}435.57$ ft. For $r = 0$, $\sigma = 3{,}669 \times 10^{-12}$m. When $2hv^2 - gr^2 = 0$ (*i. e.*, $\theta = \pi/2$) $\sigma = 1{,}612 \times 10^{-13}$ m. When $R = 6{,}604$, $\sigma = 1{,}823 \times 10^{-11}$ m. In the last instance $r \doteq 37{,}435.0001$ ft. If the entire mass were distributed uniformly over the maximum circle the surface density would be approximately $2{,}271 \times 10^{-13}$ m.

MECHANICS.

296. Proposed by C. N. SCHMALL, New York City.

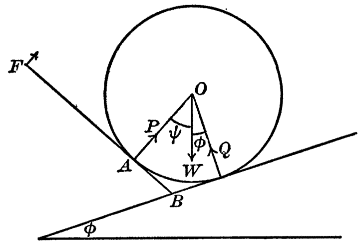
A force F is exerted in moving a horizontal cylinder up an inclined plane by means of a crowbar of length l. If R be the radius of the cylinder, W its weight, φ the inclination of the plane to the horizontal and ψ the inclination of the crowbar to the horizon, show that

$$F = \frac{WR\sin\varphi}{l[1 + \cos(\varphi + \psi)]}.$$

SOLUTION BY A. M. HARDING, University of Arkansas.

The cylinder is in equilibrium under the three forces P, Q, and W. Hence, by Lami's Theorem, $P : Q : W = \sin\varphi : \sin\psi : \sin(\varphi + \psi)$.

$$\therefore \quad P = \frac{W \sin\varphi}{\sin(\varphi + \psi)}, \qquad Q = \frac{W \sin\psi}{\sin(\varphi + \psi)}.$$



From the triangle AOB we obtain

$$AB = R \tan\frac{\varphi + \psi}{2} = R\frac{\sin(\varphi + \psi)}{1 + \cos(\varphi + \psi)}.$$

Taking moments about B we obtain

$$F \times l = P \times AB = \frac{WR \sin\varphi}{1 + \cos(\varphi + \psi)};$$

whence

$$F = \frac{WR \sin\phi}{l[1 + \cos(\phi + \psi)]}.$$

Solved similarly by CLIFFORD N. MILLS.

NUMBER THEORY.

210. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

If a and b are relatively prime and $(a + b)$ is even, then $(a - b)ab(a + b) \equiv 0$, (mod. 24).

SOLUTION BY R. M. MATHEWS, Riverside, California.

The statements a and b relatively prime and $a + b$ even imply

$$a = 2m + 1,\ b = 2n + 1$$

and

$$a + b \equiv 0 \ (\text{mod. } 2),$$

$$a - b \equiv 0 \ (\text{mod. } 2).$$

If m and n both odd or both even,

$$a - b \equiv 0 \ (\text{mod. } 4).$$

If m and n be one odd and the other even,

$$(a + b) \equiv 0 \ (\text{mod. } 4).$$